Average trajectory of returning walks

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(Received 30 June 2004; revised manuscript received 5 February 2004; published 30 April 2004)

We compute the average shape of trajectories of some one-dimensional stochastic processes $x(t)$ in the (t,x) plane during an excursion, i.e., between two successive returns to a reference value, finding that it obeys a scaling form. For uncorrelated random walks the average shape is semicircular, independent from the single increments distribution, as long as it is symmetric. Such universality extends to biased random walks and Levy flights, with the exception of a particular class of biased Levy flights. Adding a linear damping term destroys scaling and leads asymptotically to flat excursions. The introduction of short and long ranged noise correlations induces nontrivial asymmetric shapes, which are studied numerically.

DOI: 10.1103/PhysRevE.69.041105 PACS number(s): 05.40.−a, 75.60.Ej, 05.45.Tp

I. INTRODUCTION

Many disordered systems respond to external solicitations by producing noise with power-law features, that can be modeled in terms of avalanches. A notable example of such phenomena is the Barkhausen effect, first observed about a century ago by recording the noise produced by the reversal of large domains in a ferromagnet. The Barkhausen noise has been incessantly investigated, both because of its practical application as a nondestructive method to test magnetic materials, and because of its conceptual relevance for the understanding of the magnetization dynamics on a microscopic scale [1]. Experiments show that both the size and the duration of avalanches of spin reversal are power law distributed over several decades. The exponents characterizing these power laws are often used to identify universality classes [2,3]. Recently, the average pulse shape has been proposed as a sharper tool for discriminating among universality classes and to test models against experiments [4]. This analysis has revealed some weaknesses of present models, since they all fail to reproduce the avalanche shapes observed experimentally. Namely, all models proposed so far produce symmetric shapes, while leftward skewed forms are observed in experiments, indicating that our understanding of the Barkhausen effect is, at the present stage, incomplete. This open issue has been the inspiration of this work.

We consider the problem of finding the average shape of a generic stochastic signal during an excursion, i.e., between two successive returns to a reference value. We hope that a deeper understanding of how the statistical properties of the signal are reflected on the shape of the average excursion can in general give insight into the understanding of the process generating the signal. In the case of Barkhausen noise, this may help identifying which crucial ingredient is missing in the theory, and lead to the introduction of more accurate models.

Beyond its interest for what concerns the understanding of Barkhausen noise, the nontrivial phenomenology of the avalanche shape leads to more general and interesting questions: What are the physical ingredients that determine the shape of the average excursion in a generic stochastic process? Are there universality classes? Does this shape encode pieces of information about the underlying physical system, which are not accessible by considering other observables? These issues have not been addressed so far. In this paper we begin a systematic investigation of the shape of the average excursion, by considering some simple stochastic processes, both uncorrelated and correlated, and with generically distributed increments. In this way we provide a first theoretical framework that may be of help in the analysis of real time series in many contexts.

In Sec. II we introduce the concept of excursion, the types of processes that we will consider in the following and the general scaling form of the average excursion. Section III presents the results for processes with uncorrelated increments (Brownian motion, random walk, Levy flights) and Sec. IV discusses the effect of a damping term in a Brownian motion. Sections V and VI consider, respectively, the effect of long- and short-ranged noise correlations. Section VII presents some conclusions and an outlook. A short account of some of the results presented here appeared in Ref. [5].

II. DEFINITION OF THE AVERAGE EXCURSION

Let us first define the average excursion of a stochastic process. We consider a real valued $1d$ process $x(t)$ defined by a Langevin equation with suitable initial conditions. An excursion of the process is the trajectory in the (t,x) plane, followed until the first return to the initial value $x(0)$ [see Fig. 1]. We are interested in the statistics of positive excursions of a given duration T, i.e., those such that $x(t)>x(0)=0$ for $0<t<T$. In particular we will denote the average excursion as $\langle x(t)\rangle_T$.

When analyzing real experimental data, one may need to extend our definition to a generic reference value different from zero. For example, this is the case for positive signals, for which the identification of excursion (avalanches) is

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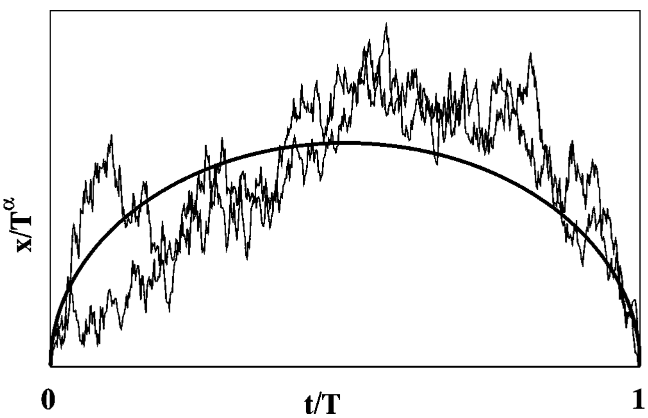

FIG. 1. Schematic representation of the average shape of a fluctuation. Thin solid lines are two realizations of the stochastic process $x(t)$, both returning for the first time at zero at time T. The thick solid line is the average shape computed over many realizations.

made unclear by the presence of background noise: one has to choose a recipe to decide when an avalanche starts or ends. In practice, one sets a threshold which is small enough not to change the shape of the avalanche, but high enough with respect to background noise.

A generic reference value a is taken into account by a translation of the origin of t and x: the excursion of $x(t)$ with respect to a is the excursion of a process $x'(t')=x(t'+t_a)-a$ with respect to $x'(0)=0$, where $x(t_a)=a$.

Notice that in general the probability distribution of first return times $P(T)$ depends on the threshold value a. For this reason we expect that variations of a will affect the average excursion $\langle x(t)\rangle_T$. In fact, for most of the simple processes discussed below the choice of a has no effect on the shape of the average fluctuation.

We will analyze two kinds of processes. The first are processes of the type

$$\partial_t x(t) = \xi(t) \tag{1}$$

with random increments $\xi(t)$ extracted from a distribution $Q(\xi)$. In Sec. III the noise ξ will be taken to be uncorrelated while in Secs. V and VI we will deal with the effect of correlations. The case considered in Sec. IV is the simplest instance of a wide class of processes, Brownian motions in a potential

$$\partial_t x(t) = -dV(x)/dx + \xi(t). \tag{2}$$

For analytical calculations it is useful to express the average excursion more explicitly. Let us first introduce the excursion distribution $\Omega(x,t|x_0,0;x_0,T)$, which is the probability that a trajectory, started in x_0 for $t=0$ and returning to x_0 for the first time at time T, is in x at time t. For each time t, Ω is the distribution of the quantity whose average is $\langle x(t)\rangle_T$, i.e.,

$$\langle x(t)\rangle_T = \frac{\int_0^\infty dx\, x\Omega(x,t|x_0,0;x_0,T)}{\int_0^\infty dx\Omega(x,t|x_0,0;x_0,T)}. \tag{3}$$

Note that $\Omega(x,t|x_0,0;x_0,T)$ is related to the distribution of first return times of the process $P(T) \equiv \int_0^\infty dx\Omega(x,t|x_0,0;x_0,T)$.

In the case of Markovian processes, as Eq. (2), Ω can be written in terms of the function $c(x,t|x_0,t_0)$, which is the probability that the process started at x_0 at time t_0 is in x at time t, with the condition that $x>0$ for all $t_0<t'<t$. The probability $\Omega(x,t|x_0,0;x_0,T)$ of the whole trajectory is the product of the probability $c(x,t|x_0,0)$ of going from x_0 to $x(t)$ times the probability $c(x_0,T|x,t)$ to go from $x(t)$ back to x_0 at time T. Notice that instead of starting exactly from 0 we have to consider the path starting and arriving in $x_0\to 0$ since c vanishes identically for $x_0=0$. Hence, for any Markovian process, we can write

$$\langle x(t)\rangle_T = \lim_{x_0\to 0^+} \frac{\int_0^\infty dx\; c(x,t|x_0,0)\; x\; c(x_0,T|x,t)}{\int_0^\infty dx\; c(x,t|x_0,0)c(x_0,T|x,t)}. \tag{4}$$

Equation (4), together with translational invariance and the scaling assumption

$$c(x,t|x_0,0) = t^\beta h[(x-x_0)/t^\alpha] \tag{5}$$

for the conditional probability $c(x,t|x_0,0)$ implies

$$\langle x(t)\rangle_T = T^\alpha f(t/T). \tag{6}$$

In some of the cases that we will consider Eq. (4) cannot be applied, since noise correlations break the Markovian property. Nevertheless, we will always find Eq. (6) to be true provided that the distribution $P(T)$ of first return times decays algebraically. In all the cases considered the exponent α coincides with the wandering exponent of the unconstrained process, defined by $\langle [x(t)-x(0)]^2\rangle \simeq t^\alpha$.

III. UNCORRELATED PROCESSES

A. Brownian motion

The simplest process is the uncorrelated Brownian motion

$$\partial_t x(t) = \xi(t), \tag{7}$$

where $\xi(t)$ is a Gaussian white noise with $\langle \xi(t)\rangle=0$ and $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$.

The probability $c(x,t|x_0,0)$ can be computed via the image method [6] as a linear combination of two solutions of the Fokker-Planck equation associated to the free process

$$c(x,t|x_0,0) = \frac{2}{\sqrt{2\pi t}}[e^{-(x-x_0)^2/(2t)} - e^{-(x+x_0)^2/(2t)}], \tag{8}$$

yielding, in the limit of small x_0

$$c(x,t|x_0,0) = \frac{2}{\sqrt{2\pi}t^{3/2}} x x_0 e^{-x^2/(2t)}. \quad (9)$$

The time-reversal and time-translational invariances of the process imply $c(x_0,T|x,t)=c(x,T-t|x_0,0)$, hence the distribution Ω is

$$\Omega(x,t|x_0,0;x_0,T) = c(x,t|x_0,0)c(x,T-t|x_0,0) \propto [(T-t)t]^{-3/2}(xx_0)^2 e^{-x^2\{1/(2t)+1/[2(T-t)]\}}. \quad (10)$$

Expression (9), inserted into Eq. (4), gives for the average excursion

$$\langle x(t)\rangle_T = T^{1/2}\sqrt{\frac{8}{\pi}}\sqrt{\frac{t}{T}\left(1-\frac{t}{T}\right)}. \quad (11)$$

The average excursion of Brownian motion is thus of the scaling form (6), with the exponent $\alpha=1/2$ coinciding with the wandering exponent of the free process and a scaling function proportional to a semicircle

$$f_U(s) = \sqrt{\frac{8}{\pi}}\sqrt{s(1-s)}, \quad (12)$$

where $s=t/T$. This result had already been noticed by Fisher [7]. The variance of the excursion is also readily computed

$$\langle [x-\langle x(t)\rangle_T]^2\rangle_T = T\left(3-\frac{8}{\pi}\right)s(1-s). \quad (13)$$

The previous results are easily generalized to the case of a Brownian motion with bias, that is, Eq. (7) with $v\equiv\langle\xi\rangle>0$. The process is now invariant under time reversal only provided the velocity is also reversed. Therefore, in this case $c(x_0,T|x,t;v)=c(x,T-t|x_0,0;-v)$. Such quantities can again be computed via the image method [8]

$$c(x,t|x_0,0;v) = \frac{1}{\sqrt{2\pi t}} e^{-(x-x_0-vt)^2/(2t)}(1-e^{2xx_0/t}). \quad (14)$$

Inserting this expression into the formula for the excursion distribution it turns out that Ω is the same of the unbiased case except for an additional factor $e^{-v^2T/2}$. Such a constant appears both in the numerator and the denominator of Eq. (4) implying that the average excursion is exactly the same of the unbiased Brownian motion (11).

Notice that the addition of a bias introduces a characteristic time of order $1/v^2$, which reflects in a cutoff in the distribution of first return times [9]

$$P(T) \propto T^{-3/2}e^{-v^2T/2}. \quad (15)$$

However, a bias does not alter the shape of the excursion: the number of trajectories that survive up to a time $T\gg 2/v^2$ is exponentially small, but the average shape of these unlikely events is exactly the same as for the unbiased case.

This is the first example of the insensitivity of $\langle x(t)\rangle_T$ with respect to changes in the distribution of single steps, a feature that will turn out to be quite generic.

B. Random walk

The Brownian motion is a continuous process in space and time. On the basis of the central limit theorem it is reasonable to expect the form of the average excursion to be the same for all processes with finite variance of the single increments. To support this conjecture we now compute $\langle x(t)\rangle_T$ for a process with finite variance, discrete in space and time, a random walk with bimodal distribution of the noise, i.e., $Q(\xi)=(\delta_{\xi,1}+\delta_{\xi,-1})/2$. The number of paths starting in 0 at time 0 and ending in x at time t without ever touching the $x=0$ axis is given by [6]

$$F(x,t) = \frac{x}{t}\frac{t!}{\left(\frac{t+x}{2}\right)!\left(\frac{t-x}{2}\right)!}. \quad (16)$$

Hence the probability to find the walker in x at time t with the condition that it has never touched the axis is obtained from Eq. (16) dividing $F(x,t)$ by $M(t)=\Sigma_x F(x,t)$, the total number of possible trajectories of t steps in the positive x half plane

$$c(x,t) = \frac{F(x,t)}{M(t)}. \quad (17)$$

Using time-reversal symmetry, the average excursion is then given by

$$\langle x(t)\rangle_T = \frac{\sum_x x\, c(x,t)c(x,T-t)}{\sum_x c(x,t)c(x,T-t)} = K\sum_{x=0}^{t} x\, c(x,t)c(x,T-t) \quad (18)$$

with $K=M(t)M(T-t)/F(1,T-1)$, where we used the fact that $\Sigma_x F(x,t)F(x,T-t)$ is independent from t and equal to $F(1,T-1)$. Introducing the variables $\gamma=x/\sqrt{t}$, $s=t/T$, and $\phi=\sqrt{s/(1-s)}$, and using the expansion $n!/(n/2)! \simeq \sqrt{2/\pi}\,2^n/n$, we get $M(t)c(\gamma\sqrt{t},t)=F(\gamma\sqrt{t},t) \simeq (2^t/\sqrt{\pi t})\gamma e^{-\gamma^2/2}$, $F(1,T-1)\simeq 2^T/\sqrt{2\pi}T^{3/2}$, and $K \simeq \sqrt{2/\pi}(T^{-1/2}/s(1-s))$ so that the average excursion is

$$\langle x(t)\rangle_T = Kt\phi\int_0^{\sqrt{t}} d\gamma\gamma^3 e^{-(\gamma/2)(1+\phi^2)} \simeq \frac{2K\phi t}{(1+\phi^2)^2}, \quad (19)$$

where in the last step the integral has been extended to infinity. Expressing ϕ in terms of $s=t/T$ we recover

$$\langle x(t)\rangle_T \simeq T^{1/2}\sqrt{\frac{8}{\pi}}\sqrt{s(1-s)} \quad (20)$$

exactly as in the continuous case.

In the case of a biased walk, where $Q(\xi)=[q\delta_{\xi,1}+(1-q)\delta_{\xi,-1}]$, one must replace the number of trajectories $F(x,t)$ with their probability: each trajectory reaching x at time t is weighted by a factor $q^{(t+x)/2}(1-q)^{(t-x)/2}$. The trajectory leading back to zero has instead a weight $q^{(T-t-x)/2}(1-q)^{(T-t+x)/2}$. The product of these weights gives simply a constant factor $[q(1-q)]^{T/2}$ both in the numerator and in the denominator of

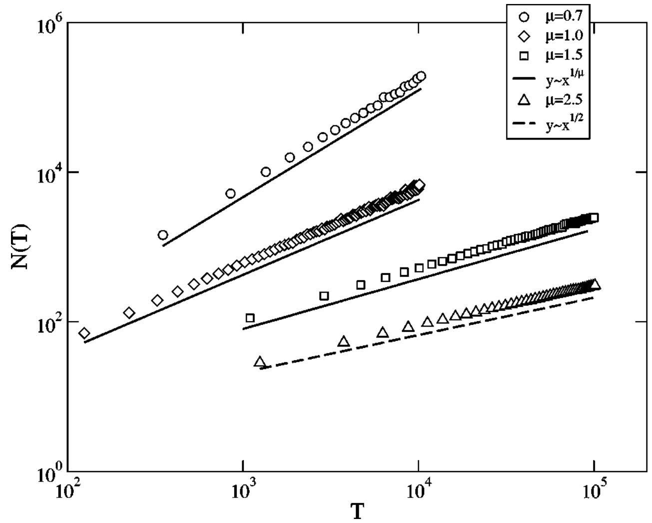

FIG. 2. The factor $N(T)$ vs T for Levy flights with several values of μ. Notice that the exponent is $1/2$ for $\mu \geqslant 2$.

formula (18). Hence the introduction of a bias does not change all moments of the distribution Ω of the excursion.

C. Unbiased Levy flights

Levy flights are statistical processes of the type (7), where the distribution of single steps $Q(\xi)$ has a fat tail decaying as $|\xi|^{-\mu-1}$ with $0<\mu<2$ so that their variance is infinite. The standard form of the central limit theorem does not hold for Levy flights: the invariant distributions under summation are the Levy stable distributions [10]. It is therefore natural to wonder whether Levy flights belong to a different universality class also with respect to the average excursion.

The analytical evaluation of $\langle x(t)\rangle_T$ for this case is not straightforward as for the processes discussed so far. While Eq. (4) still holds, the image method cannot be used to determine $c(x,t|x_0,0)$, because the Fokker-Planck equation for the free process is not local [11]. We have therefore computed the average excursion numerically, considering steps performed at discrete integer times with absolute value distributed according to $Q(|\xi|) \propto (|\xi|+1)^{-\mu-1}$ and random sign.

Details about the evaluation of $\langle x(t)\rangle_T$ in this and in the other cases where numerical results have been obtained are presented in Appendix A.

To check the validity of the scaling hypothesis we compute the quantity $N(T)$, such that $\langle x(t)\rangle_T = N(T) f(t/T)$, where we choose, with no loss of generality to normalize the scaling function f so that $\int_0^1 f(s)ds=1$. If scaling holds, $N(T)$ has to be proportional to T^α and the normalized average shapes $\langle x(t)\rangle_T / N(T)$ must collapse on the same curve for different T. In Fig. 2 we plot $N(T)$ for several values of μ, showing that $\alpha = \max[1/2, 1/\mu]$. Also in this case α coincides with the wandering exponent of the free process, which is $1/\mu$ for $\mu<2$ and is the usual diffusive one when $\mu \geqslant 2$ [12].

Figure 3 reports the shape of the average excursion for values of μ such that the second or even the first moment of the single step distribution is infinite. In all cases the curves for different values of T collapse and the scaling form is exactly the same as of the Brownian motion: The shape of the average excursion is completely independent from the distribution of single steps.

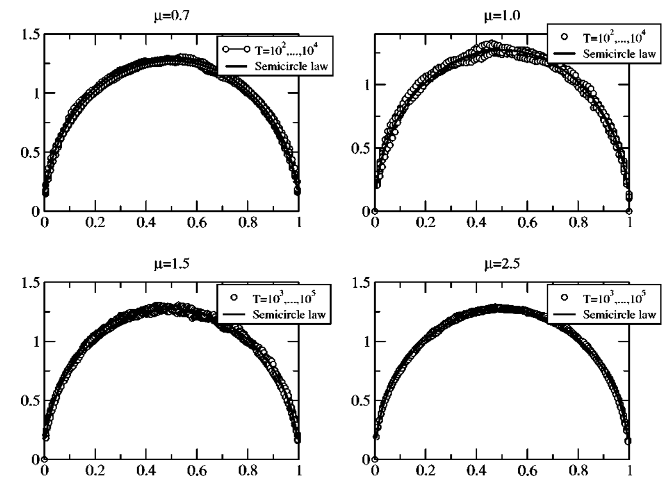

FIG. 3. Scaling function f for for Levy flights with several values of μ.

In Appendix B we report the results also for the variance of Levy flights. We have not been able to prove this result analytically for Levy flights. However, in the case $\mu=1$, we have considered the Levy-stable distribution, the Cauchy distribution

$$Q(\xi) = \frac{1}{\pi(1+\xi^2)}. \tag{21}$$

We have computed numerically the probability density $c(x,t)$ for such a process. The result is presented in Fig. 4, where it is compared with the ansatz

$$c(x,t) = \frac{a}{t}\frac{\sqrt{x/t}}{1+(x/t)^{5/2}}, \tag{22}$$

where a is a normalization constant. Formula (22) is the simplest expression that interpolates between the small s and

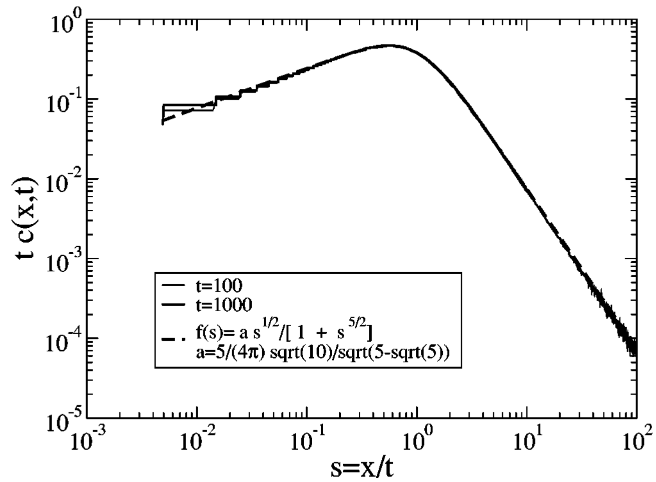

FIG. 4. Comparison between $c(x,t)$ evaluated numerically for for Levy flights with steps distributed according to Eq. (21) and the ansatz (22).

large s power-law behaviors found numerically. The agreement between the numerical results and the formula is striking. A different ansatz has been checked to be false for such a case, in Ref. [18].

Inserting the expression (22) into Eq. (4) and performing the integrals, one obtains

$$\langle x(t)\rangle_T = T\sqrt{s(1-s)} \tag{23}$$

adding another piece of evidence to the universality of $f(s)$.

D. Biased Levy flights

We now consider the case of biased Levy flights, where single increments are distributed symmetrically as in the unbiased case plus a constant term v. In this case the average form of the excursion is in general asymmetric for finite times (toward left or right depending on the sign of v). To understand this behavior, it is fundamental to consider the relative importance in the equation of motion of the drift term and of the wandering due to the stochastic variable ξ.

The former is clearly vT, while the latter grows as $T^{1/\mu}$. Hence a crossover time $T^*(v)\sim v^{-\mu/(1-\mu)}$ exists between two regimes: which of the two mechanisms dominates depends on the value of μ. For $1<\mu<2$ the bias dominates for large times, while the wandering is larger than vT for $T\ll T^*$. For $\mu<1$ the opposite is true and asymptotically the bias does not play any role.

When the bias is irrelevant, the behavior is the same as of the unbiased case: $\langle x(t)\rangle_T$ is given by Eq. (12), $\alpha=1/\mu$ and the first passage time distribution $P(T)$ decays as $T^{-1-1/\mu}$ [13].

When the bias dominates, instead, the shape of the excursion is completely different. In this case the trajectories are practically deterministic, i.e., ballistic motions with velocity v. However, the noise term is crucial to have the constraint $x(T)=0$ satisfied, since at $t\simeq T$ a very large fluctuation is needed. As a consequence, the distribution of return times is $P(T)\sim T^{-1-\mu}$, the exponent α is 1 and the average excursion has a triangular shape $f_T(s)=2s$. This bias-dominated regime is shown for $\mu=1.5$ and $T\gg T^*$ in Fig. 5. The crossover between the two asymptotic regimes is very slow and it is not possible to run a single simulation long enough to exhibit the full transition between the early and late regimes. Nevertheless it is possible to distinguish clearly between the behavior for $\mu>1$ and $\mu<1$. In the former case (Fig. 6) the form of the average excursion becomes more and more skewed with time, while in the latter case (Fig. 7) the opposite behavior is observed.

To give a measure of the asymmetry, we consider the quantity

$$\int_0^1 ds f(s)\mathrm{sgn}\ (s-1/2), \tag{24}$$

which is zero for the semicircle and $1/2$ for the triangle. The insets of Figs. 6 and 7 clearly indicate that, for $\mu=1.5$, it grows, while it decreases to zero for $\mu=0.5$, consistently with the argument presented above.

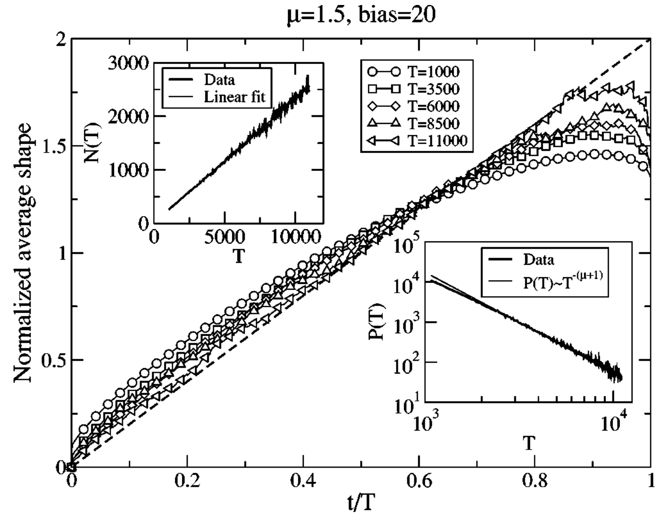

FIG. 5. Bias-dominated regime for Levy-distributed increments ($\mu=1.5$) with bias $v=20$. Main: Normalized scaling function f converging toward the asymptotic form $f(s)=2s$. Upper inset: The factor $N(T)$ growing as T^α with $\alpha=1$. Lower inset: First return distribution $P(T)$ decaying as $T^{-\mu-1}$.

Therefore, we can conclude that, while in the unbiased case the average excursion of a generic uncorrelated stochastic process with symmetric steps obeys asymptotically the scaling form (6) with universal shape $f_U(s)$ [Eq. (12)], in the biased case, this is true only for $\mu<1$ or $\mu\geq 2$. For $1<\mu<2$ instead, the presence of a bias leads to an asymptotic average excursion with triangular shape. The results for biased flights are summarized in Fig. 8 [14].

IV. DAMPED BROWNIAN MOTION

We now deal with a Brownian motion in a potential, Eq. (2). We treat only the simplest possible case, an harmonic

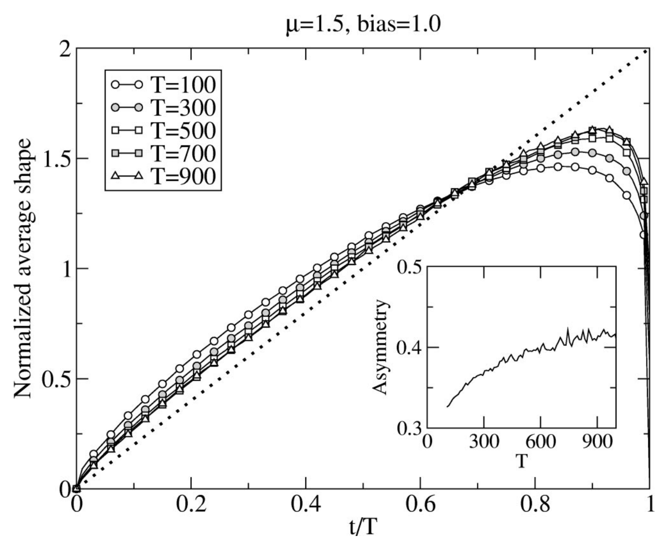

FIG. 6. Main: Scaling function f for biased Levy flights with $\mu=1.5$ and $v=1$. Inset: Temporal evolution of the asymmetry parameter [Eq. (24)].

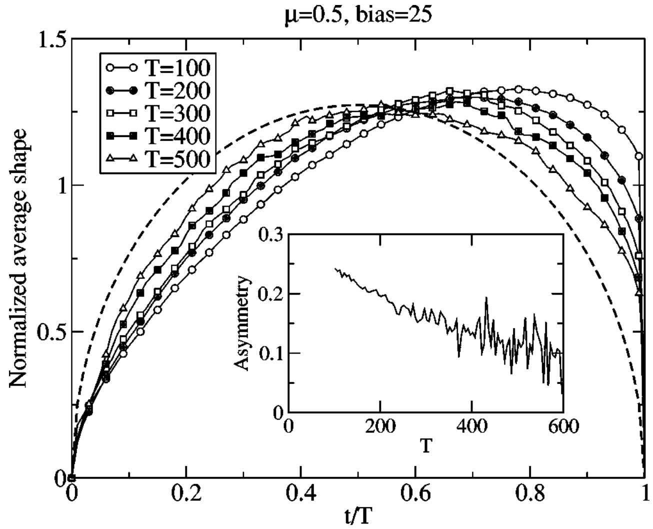

FIG. 7. Main: Scaling function f for biased Levy flights with $\mu=0.5$ and $v=25$. Inset: Temporal evolution of the asymmetry parameter [Eq. (24)].

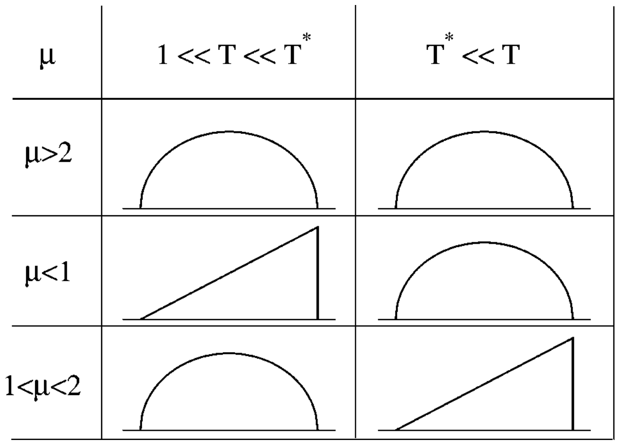

FIG. 8. Sketch of the average shape for positively biased flights. The crossover time T^*, depending on the value of the bias, changes the asymptotic behavior only for $1<\mu<2$. For $\mu<1$ the average trajectory is asymmetric in the intermediate regime $T \ll T^*$, and it takes a triangular shape in the limit $T\to\infty$ after $T^*\to\infty$.

potential $V(x)=\lambda x^2$, so that the Brownian motion is pushed toward the origin by a linear damping term

$$\partial_t x(t) = -\lambda x + \xi(t). \tag{25}$$

The formal solution of this equation is given by

$$x(t) = x_0 e^{-\lambda t} + \int_0^t ds e^{-\lambda(t-s)} \eta(s). \tag{26}$$

Since $x(t)$ is linearly related to $\eta(t)$, it has a Gaussian distribution characterized by its first and second moment, that can be obtained by averaging over the noise:

$$m(t) = \langle x(t) \rangle = x_0 e^{-\lambda t},$$
$$\sigma_\lambda^2(t) = \langle [x(t) - \langle x(t) \rangle]^2 \rangle = \frac{1-e^{-2\lambda t}}{2\lambda}. \tag{27}$$

The normalized probability density for the free process is therefore

$$P(x,t) = \frac{1}{\sqrt{2\pi}\sigma_\lambda(t)} \exp\left[\frac{(x-x_0e^{-\lambda t})^2}{2\sigma_\lambda^2(t)}\right]. \tag{28}$$

One can easily check that $P(x,t)$ solves the Fokker-Planck equation

$$\partial_t P(x,t) = \tfrac{1}{2}\partial_x^2 P(x,t) - \lambda \partial_x[xP(x,t)] \tag{29}$$

associated with the Langevin equation (25).

The distribution of first return times is

$$P(T) \propto \frac{1}{8\pi}\frac{e^{2\lambda T}}{\sigma_{-\lambda}^3(T)} = \frac{1}{8\pi}(2\lambda)^{3/2}\frac{e^{-\lambda T}}{(1-e^{-2\lambda T})^{3/2}}. \tag{30}$$

Thus

$$P(T) \propto \begin{cases} \dfrac{8}{\pi}T^{-3/2}, & T \ll 1/\lambda \\ \dfrac{8}{\pi}(2\lambda)^{3/2}e^{-\lambda T}, & T \gg 1/\lambda, \end{cases} \tag{31}$$

For $T \ll 1/\lambda$, the $T^{-3/2}$ behavior characteristic of Brownian motion is recovered. When T is of order $1/\lambda$, the power-law behavior is cut off exponentially.

To calculate the average excursion we need to evaluate the probability $c(x,t|x_0,0)$ that a walk originating at x_0 at time 0 is found in x at time t, without having ever touched the origin. Since Eq. (25) is linear, the image method can be applied, yielding

$$c(x,t|x_0,0) = \frac{1}{\sqrt{2\pi}\sigma_\lambda(t)}\left\{\exp\left[-\frac{(x-x_0e^{-\lambda t})^2}{2\sigma_\gamma^2(t)}\right] - \exp\left[-\frac{(x+x_0e^{-\lambda t})^2}{2\sigma_\lambda^2(t)}\right]\right\}. \tag{32}$$

In the limit $x_0 \to 0$

$$c(x,t|x_0,0) \propto \frac{2x_0 x e^{-\lambda t}}{\sqrt{2\pi}\sigma_\lambda^3(t)} e^{-[x^2/2\sigma_\lambda^2(t)]} \tag{33}$$

The quantity $c(x_0,T|x,t)$ for $x_0\to 0$ is obtained in a similar way

$$c(x_0,T|x,t) \propto \frac{2x_0 x e^{-\lambda(T-t)}}{\sqrt{2\pi}\sigma_\lambda^3(T-t)} e^{-[x^2 e^{-2\lambda(T-t)}/2\ \sigma_\lambda^2(T-t)]}. \tag{34}$$

Defining $\tilde{\sigma}_2^\lambda(t)=\sigma_\lambda^2(t)e^{2\lambda t}$ one can rewrite Eq. (34) as

$$c(x_0,T|x,t) \propto \frac{2x_0 x e^{2\lambda(T-t)}}{\sqrt{2\pi}\tilde{\sigma}_3^\lambda(T-t)} e^{-[x^2/2\ \tilde{\sigma}_\lambda^2(T-t)]}. \tag{35}$$

Note that $\tilde{\sigma}_\lambda^2(t)=\sigma_\lambda^2(t)e^{2\lambda t}=\sigma_{-\lambda}^2(t)$, thus the process with reversed time formally corresponds to the process with $\lambda\to-\lambda$.

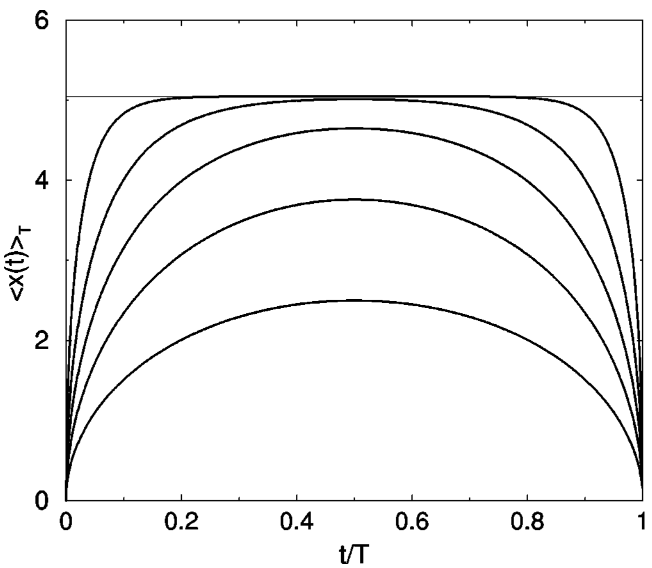

FIG. 9. Average excursion for a damped random walk [Eq. (38)] for $1/\lambda = 20$. From top to bottom lines are for $T = 10, 25, 50, 100, 250$. Notice that the shape flattens to the constant value $\sqrt{4/(\pi\lambda)}$ (thin line).

The distribution of excursions Ω is then

$$\Omega(x,t|x_0,0;x_0,T) \propto e^{-\lambda T}[\sigma_\lambda(t)\sigma_{-\lambda}(T-t)]^{-3} e^{-x^2/[2\sigma_{eq}^2(t,T)]}, \tag{36}$$

where the variance of the Gaussian factor is

$$\sigma_{eq}^2 = [\sigma_\lambda^{-2}(t) + \sigma_{-\lambda}^{-2}(T-t)]^{-1} = \frac{1}{2\lambda}\left[\frac{(1-e^{-2\lambda t})(1-e^{-2\lambda(T-t)})}{1-e^{-2\lambda T}}\right]. \tag{37}$$

Inserting Eqs. (33) and (34) into Eq. (4) yields

$$\langle x(t)\rangle_T = \sqrt{\frac{8}{\pi}}\sigma_{eq}(t,T) = \sqrt{\frac{8}{\pi}}\frac{1}{\sqrt{2\lambda}}\sqrt{\frac{(1-e^{-2\lambda t})(1-e^{-2\lambda(T-t)})}{1-e^{-2\lambda T}}}. \tag{38}$$

As expected the existence of a characteristic time in the problem $(1/\lambda)$ breaks down scaling; the shape of the average excursion changes with T (Fig. 9). However formula (6) still holds in the two asymptotic limits

$$\langle x(t)\rangle_T = \begin{cases} \sqrt{\frac{8}{\pi}}\sqrt{\frac{t(T-t)}{T}} & t, T-t \ll 1/\lambda \\ \sqrt{\frac{8}{\pi}}\frac{1}{\sqrt{2\lambda}} & t, T-t \gg 1/\lambda. \end{cases} \tag{39}$$

Thus in the small λ limit the semicircle is recovered, while in the large λ limit the curve flattens around a value proportional to $1/\sqrt{\lambda}$, while keeping the semicircular tails. The crossover between the two regimes corresponds to a change in the variance σ_{eq}^2 of Ω

$$\sigma_{eq}^2 = \begin{cases} \frac{t(T-t)}{T} & t, T-t \ll 1/\lambda \\ \frac{1}{2\lambda} & t, T-t \gg 1/\lambda, \end{cases} \tag{40}$$

Thus fluctuations saturate at the value $1/(2\lambda)$, indicating the absence of correlations on time scales longer than the characteristic time $1/\lambda$.

In the computation presented here we have taken the reference value to coincide with the origin, i.e., $a=0$. In the presence of a potential, the effect of a reference value different from zero corresponds to the change $V(x) \to V(x+a)$. For the damped Brownian motion, this is equivalent to the introduction of a bias λa. In general, such a bias perturbs the form of the average excursion, at odds with the case of free random walks, which are unchanged by the presence of a bias. However, the change in the average shape is small provided a is small compared with $4/\sqrt{\pi\lambda}$, the maximal amplitude of $\langle x(t)\rangle_T$.

V. LONG-RANGED CORRELATIONS

We now start considering the effect of the introduction of temporal correlations in the process. In particular, we study processes of the form $\partial_t x(t) = \xi(t)$, with correlations between single increments $g(t,t') \equiv \langle \xi(t)\xi(t')\rangle - \langle \xi(t)\rangle\langle \xi(t')\rangle \neq \delta_{t,t'}$.

We first focus on a process $\partial_t x(t) = \xi(t)$ where the noise performs in its turn a Brownian motion $\partial_t \xi(t) = \eta(t)$ with $\langle \eta\rangle = 0$ and $\langle \eta^2\rangle = 1$. Clearly this process is non-Markovian and can be written as the random accelerated particle (RAP),

$$\partial_t^2 x(t) = \eta(t). \tag{41}$$

The correlation function of $\xi(t)$, $g(t,t') = \min(t,t')$ does not decay to zero when $t-t'$ diverges: the noise has then infinitely ranged correlations. This process has been studied recently with regards to polymers [15] and the inelastic collapse of granular matter [16].

It is important to stress here that, at odds with the previous cases, the non-Markovian nature of the process implies that to define completely a fluctuation one has also to consider the initial and final velocities (the process is Markovian if one considers the broader space of coordinate and velocity). We consider separately avalanches beginning with zero or a finite velocity, that will yield different results. For what concerns the final velocity the most natural thing is to average over all final velocities. The condition on the initial velocity is numerically very easy to implement, since it corresponds to setting $\eta(0) = v_0$. Instead, the condition on the final velocity is more delicate. Considering returns within a strip $(-\epsilon, +\epsilon)$ implies, when using discrete times, a hidden condition on the final velocities. This is apparent if one considers the long time decay of the distribution of first return times. If one considers discrete times and a return in a strip, the known result $P(T) \sim T^{-5/4}$ [17] is not recovered. The expected exponent is found instead if one averages over all trajectories positive up to $t < T$ and becoming negative for $t = T$. Therefore, we considered only this latter kind of trajec-

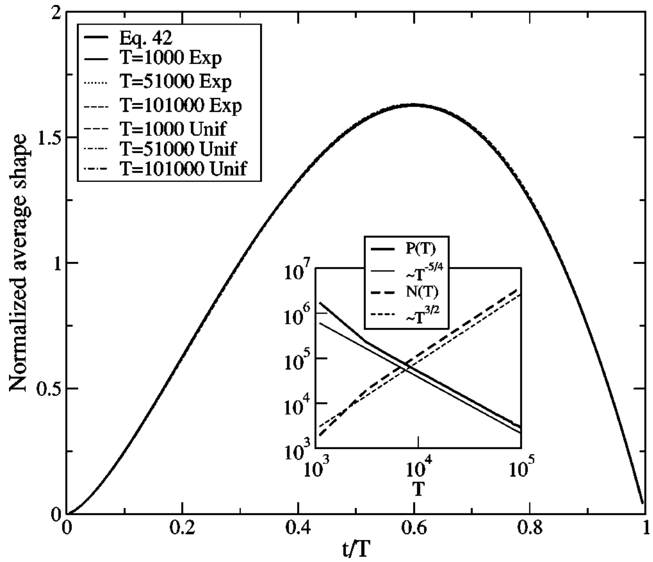

FIG. 10. Main: Average excursion for a RAP with zero initial velocity, with uniform or exponential distribution of the noise η, showing perfect agreement with the simple form (42). Inset: Factor $N(T)$ and first-return time distribution $P(T)$ for uniform noise distribution.

tories. We believe that, asymptotically, this coincides with the continuous time process averaging over all final velocities.

Figure 10 reports the average excursion shape for two different distributions of η with finite variance, one uniform and the other exponential, both in the case of zero initial velocity. It is clear that scaling holds very well for all values of T considered and for both distributions of the noise, and that the scaling function is asymmetric. As shown in the inset, the exponent α is again equal to the free wandering exponent, which is 3/2. Remarkably, the scaling function is indistinguishable from the simple form

$$f(s) = \tfrac{3}{2} s^{3/2}(1-s). \tag{42}$$

In Appendix B we report also the results for the variance of the excursion, showing that it scales as the first moment.

When the initial velocity v_0 is finite, instead, the asymptotic scaling associated with an average shape (42) is preceded by a transient regime with a more symmetric $\langle x(t)\rangle_T$, depending on T and v_0 (Fig. 11). For large v_0 the shape is very close to a simple parabola $6s(1-s)$. We have also checked that the exponent α crosses over from a ballistic value $\alpha=1$ at short times to the asymptotic value 3/2. Comparing the values of α in the two regimes we expect the crossover time to be proportional to v_0^2.

We then consider RAP with noise η distributed with slowly decaying tails $P(|\eta|) \sim \eta^{-1-\mu}$. With zero initial velocity the scaling function depends on the value of μ (Fig. 12), at odds with what occurs for uncorrelated unbiased processes. The skewness is toward right, the more so for small values μ. The exponent α is equal to $1+1/\mu$.

Despite the dependence of the detailed scaling form of $\langle x(t)\rangle_T$ on the distribution of single steps, some degree of

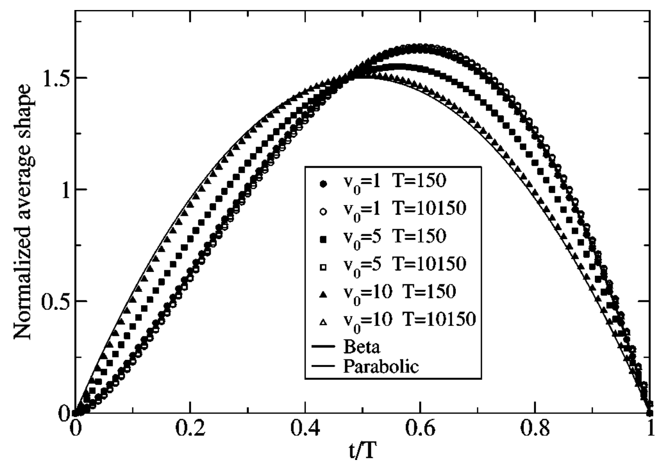

FIG. 11. Main: Scaling function f for a RAP with finite initial velocity, with uniform distribution of the noise η.

universality remains also for the RAP process. The right panel of Fig. 13 shows that the exponent characterizing the behavior of $f(s)$ for $s\to 1$ is universal, being linear also for $\mu=1$. For small values of s instead, a careful analysis indicates that the actual exponent of the power-law behavior is slightly different from the value 3/2 obtained for finite variance (Fig. 13, left panel). We do not have an explanation for the value of this exponent. However, we cannot rule out that such universality is restored for larger values of T. We have checked that, in the case of finite initial velocity v_0, the transient parabolic shapes are present also for values of $\mu<2$.

VI. SHORT-RANGED CORRELATIONS

We now turn to the case of short-ranged correlations. We consider a process with correlations decaying exponentially over an interval τ,

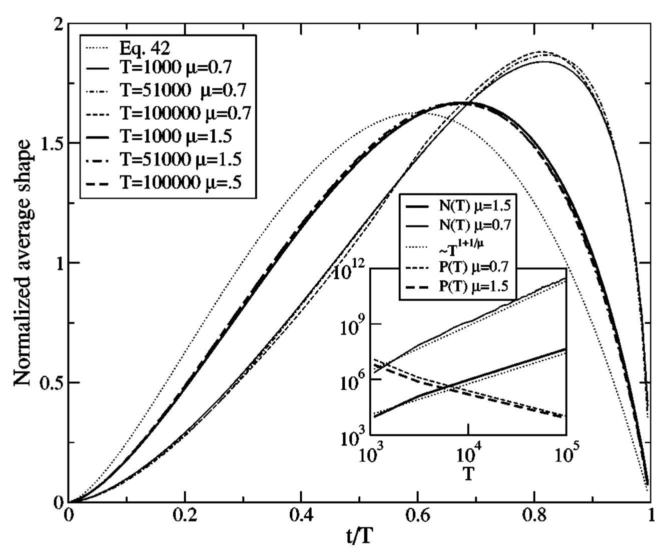

FIG. 12. Main: Scaling function f for a RAP with noise η distributed with slow decaying tails $P(|\eta|)$ and several values of μ, compared with the simple form (42). Inset: Factor $N(T)$ and first-return time distribution $P(T)$ for the same values of μ.

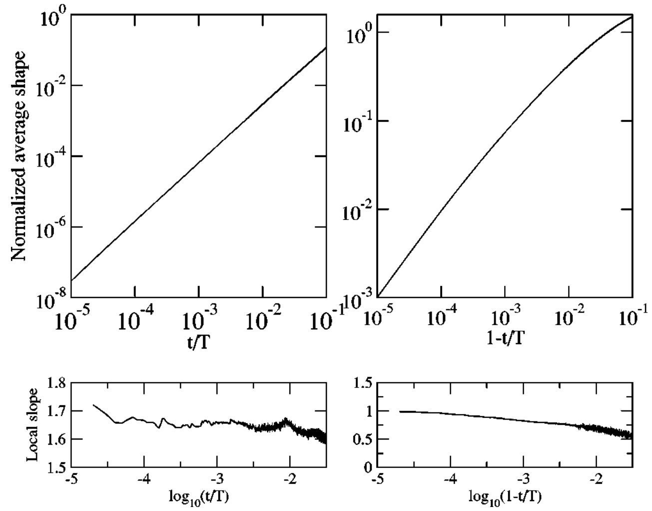

FIG. 13. Left top: Small t/T tail of the scaling function f for RAP with Cauchy distribution of single steps. Left bottom: Local effective exponent computed on the figure above. Right top: Small $1-t/T$ tail of the scaling function f for RAP with Cauchy distribution of single steps. Right bottom: Local effective exponent computed on the figure above.

$$g(t,t') = \exp\,(-|t-t'|/\tau). \tag{43}$$

A process of this type can be obtained in practice by feeding the random walk with noise obeying a damped random walk of the form $\xi(t+1)=\gamma\xi(t)+(\sqrt{1-\gamma^2})\eta(t)$ with uncorrelated η. The correlator of $\xi(t)$ is easily shown to decay exponentially, with a characteristic time $\tau=-1/\ln\gamma$.

For such a process, we do not expect scaling (6) to hold for all values of T. We can anticipate instead two regimes depending on the duration T of the trajectories considered. For short times $T\ll\tau$, noise is correlated during the whole trajectory and the behavior must be the same of the case with infinitely ranged correlations treated above. For long times $T\gg\tau$, on the contrary, noise is correlated only for intervals that are short compared to the total duration of the excursion. Hence the process is equivalent to an uncorrelated process with some effective distribution of the single increments.

Numerical results fully confirm this picture. In Fig. 14 we show the case of a short-memory process with zero initial velocity and finite variance noise. For short times $T\ll\tau$ the shape is very close to the form (42), valid for RAP. A slow crossover leads for longer times to the semicircle law valid for uncorrelated processes. We have checked that the expected pattern of behavior occurs also for nonzero initial velocity and for Levy-distributed noise.

VII. CONCLUSIONS

Let us summarize the results presented in the previous sections. We have studied the statistics of excursions in some classes of stochastic processes, with particular attention to the average shape. For uncorrelated free processes we have found that the average excursion has a scaling function proportional to a semicircle, independent from the distribution of the single steps provided it is symmetric. This holds not

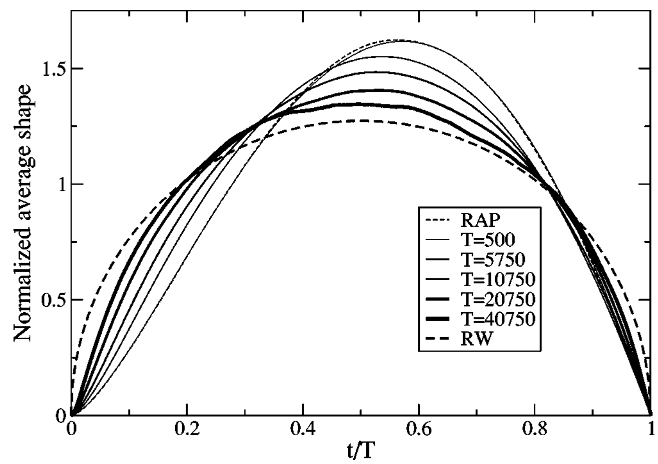

FIG. 14. Normalized average shape for a short-memory process with zero initial velocity, finite variance noise and $\tau=1000$. The dashed lines are the expected limiting curves for $T\ll\tau$ and $T\gg\tau$.

only for distributions that renormalize to the Gaussian, but also for the class of distributions that renormalize to symmetric Levy stable distributions. More generally the scaling function is unchanged when a bias is introduced, with the notable exception of the case with $1<\mu<2$, where the asymptotic shape of fluctuations is triangular. The addition of a linear damping term in the Langevin equation for the process introduces a characteristic time scale, that separates between two regimes: for short times the process is dominated by noise, and the excursion is the same as in the free case. For longer times, scaling breaks down and the shape of the average fluctuation flattens to a value independent from its duration. Furthermore, we have analyzed the effect of noise correlations for the free process. When correlations are long ranged the shape of a fluctuation depends on the initial velocity v_0. For $v_0=0$ we find that the scaling function has asymmetric tails $s^{3/2}$ and $(1-s)$ in the Gaussian case, while the situation for Levy distributed steps is less clear. For $v_0>0$ a transient regime exists such that the scaling function has linear tails (independent from the distribution of the single steps), before it crosses over to the asymptotic form which is the same of the $v_0=0$ case. Finally, in the case of short-range correlated noise, the range of correlation sets a time scale that separates between a short time regime, where, as expected the behavior is similar to the long-range case, followed by a crossover to the asymptotic uncorrelated behavior.

Application of this analysis to real data requires some care. Indeed in many situations of practical interest one deals with long time series consisting of a large number of successive fluctuations. In such a case, if one computes $\langle x(t)\rangle_T$ by averaging over successive returns to the value $x=a$ one may average over pulses that are not statistically independent.

In our work, on the contrary, we take care to average always on independent events. When the process we consider is Markovian this does not require particular prescriptions. In this case averaging over successive fluctuations in a single realization is equivalent to averaging over avalanches belonging to different realizations. Otherwise one should consider avalanches separated by times larger than the larg-

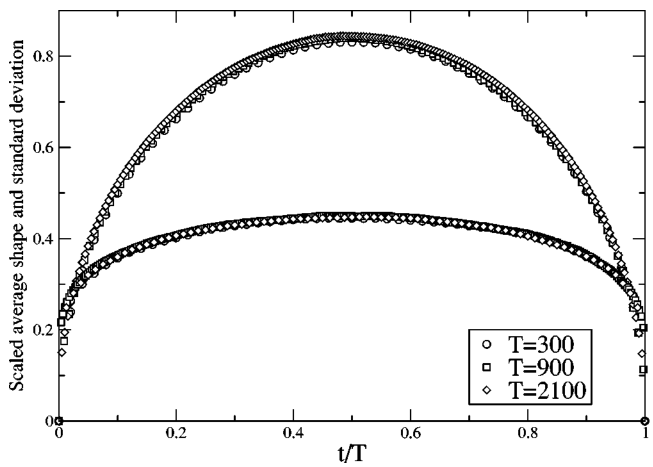

FIG. 15. Plot of the average shape and standard deviation for Levy flight with $\mu=1.5$. Curves for different durations are divided by the scaling factor $T^{1/\mu}$. The upper curves are the average shapes, the lower ones are the standard deviations.

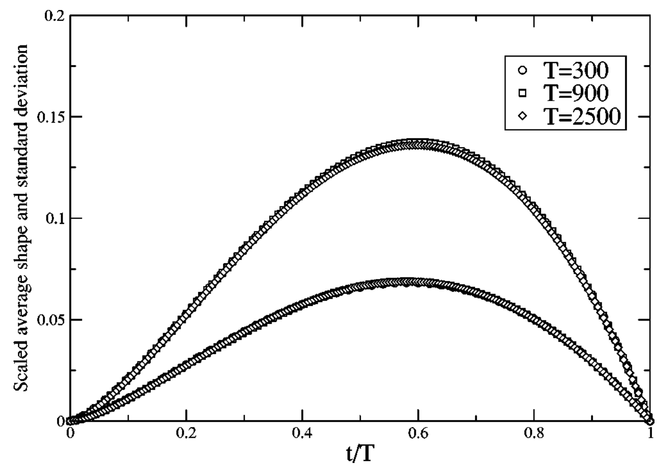

FIG. 16. Plot of the average shape and standard deviation for RAP with Gaussian noise. Curves for different durations are divided by the scaling factor $T^{3/2}$. The upper curves are the average shapes, the lower ones are the standard deviations.

est correlation time in the system, if the analysis is restricted to a single realization. If the correlation time is infinite, as in the RAP, one should consider only fluctuations belonging to independent realizations.

Another relevant issue for the application to real time series concerns the amount of events required to obtain sufficiently clean results. In principle one should average over fluctuations of exactly the same duration, and rescale afterward. This may turn out to require an exceedingly large number of events. An alternative procedure is to assume scaling and average over fluctuations of different duration, properly rescaled, with an exponent that can be obtained by plotting the size of fluctuations as a function of their duration. This also checks whether scaling holds or not.

For the case of Barkhausen noise, which was the initial inspiration of this work, we suspect that the asymmetric shape observed in experiments must be due to the presence of some kind of correlations. However, the kind of correlations that we have analyzed give rightward asymmetric shapes, while the one observed experimentally are leftward. This calls for further analysis, of more general processes.

APPENDIX A: DETAILS ABOUT THE NUMERICAL RESULTS

When performing numerical simulations, we have taken time to be discrete and space continuous, thus the concept of first return to the initial value needs some clarification. We have considered the process to return to the initial value when its value is in a small interval $[-\epsilon,\epsilon]$ around it. In order not to introduce an artificial asymmetry we have applied the same condition to the first step of the excursion as well. This is implemented by letting the process start at $x=0$ for some negative time and taking as $t=1$ the first time such that $x(t)>\epsilon$. The average is then performed over all trajectories that first return between $-\epsilon$ and ϵ at a specified time T under the constraint that $x(t)>\epsilon$ for $1<t<T$. Care has to be used when choosing the value of ϵ, which should be as small as possible to give ϵ independent results. At the same time too small values of ϵ make the numerical simulation very time consuming.

In all simulations T is integer; hence it is in principle possible to average only over trajectories that return after *exactly* T steps. However, T cannot be decided *a priori* but is the outcome of the simulation; collecting a large number of trajectories for a single large T may be a prohibitive task. Therefore, similarly to what is done in experiments, we have performed a binning procedure, by averaging over all runs that return within a time interval between $T-\Delta T$ and $T+\Delta T$. Since the relative size of the bins $2\Delta T/T$ decreases as T grows, we are sure that this procedure does not lead to significant perturbation of the results for large T.

Finally, when presenting numerical evaluations of the average excursion we plot $\langle x(t)\rangle_T$ normalized by the factor $N(T)=\int_0^1 ds\langle x(sT)\rangle_T$. This allows one to check the form (6) in two ways. If scaling holds then $N(T)$ grows as a power law (the exponent is α) and the different curves collapse on a universal shape [which is $f(s)$].

APPENDIX B: NUMERICAL COMPUTATION OF VARIANCE

For the simplest processes that can be attacked analytically we have already computed the variance:

$$\sigma_T^2(t)=\langle[x-\langle x(t)\rangle_T]^2\rangle_T.$$

In the other cases it is possible to determine numerically such a quantity. In this appendix we show, as an example, the results for two of such cases, i.e., the unbiased Levy flight with $\mu=1.5$ and the RAP. In Figs. 15 and 16 we have reported the average excursion $\langle x(t)\rangle_T$ for different durations T divided by the expected scaling factor T^α. Correspondingly we plot the values of the standard deviation $\sigma_T(t)$ divided by the same factor. One sees that σ scales exactly as the average shape in both cases.

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